MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — Quiz Date: May 9, 2002
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QUIZ 3 SOLUTIONS

PROBLEM 1: NEUTRON-PROTON RATIO AND BIG-BANG NUCLEOSYNTHESIS *(20 points)*

(a) In thermal equilibrium, the ratio of neutrons to protons is given by a Boltzmann factor,

$$\boxed{\frac{n_n}{n_p} = e^{-\Delta m\, c^2/kT}\ ,} \tag{1}$$

where $\Delta m = (m_n - m_p)$. For $\Delta m\, c^2 = 1.293 \times 10^6$ eV, $k = 8.617 \times 10^{-5}$ eV/K, and $T = 5 \times 10^{10}$ K, this gives

$$\frac{n_n}{n_p} = \exp\left\{-1.293 \times 10^6/(8.617 \times 10^{-5} \times 5 \times 10^{10})\right\} = \boxed{0.741\ .} \tag{2}$$

Caveat (for stat mech experts): The above formula would be a precise consequence of statistical mechanics if the neutron and proton were two possible energy levels of the same system. In this case one would describe the system using the *canonical ensemble*, which implies that the probability of the system existing in any specific state i is proportional to $\exp(-E_i/kT)$, where E_i is the energy of the state. However, the neutron and proton are not really different energy levels of the same system, because the conversion between neutrons and protons involves other particles as well; a sample conversion reaction would be

$$n + \nu_e \longleftrightarrow p + e^-\ ,$$

where ν_e is the electron neutrino, and e^- is the electron. This means that if the universe contained a very large density of electron neutrinos, then n-ν_e collisions would occur more frequently, and the reaction would be driven in the forward direction. Thus, a large density of electron neutrinos would lead to a lower ratio of neutrons to protons than the Boltzmann factor given above. Similarly, if the universe contained a large density of electrons, then the reaction would be driven in the reverse direction, and the ratio of neutrons to protons would be higher than the Boltzmann factor. A complete statistical mechanical treatment of this situation would use the *grand canonical ensemble*, which describes systems in which the number of particles of a given type can change by chemical reactions. In this formalism the density of each type of particle is related to a quantity called the *chemical potential* μ, where in general the relationship is given by

$$n = g \int \frac{d^3p}{(2\pi\hbar)^3}\, \frac{1}{e^{(E-\mu)/(kT)} \pm 1}\ , \tag{3}$$

where

$$E = \sqrt{|\vec{p}|^2\, c^2 + m^2 c^4}\ , \tag{4}$$

the $+$ sign holds for Fermi particles, the $-$ sign holds for Bose particles, and the factor g has the same meaning as in Lecture Notes 7. It is sometimes more convenient to use E as the variable of integration, in which case the formula above can be written as

$$n = \frac{g}{2\pi^2(\hbar c)^3} \int_m^\infty \frac{(E^2 - m^2)^{1/2}}{e^{(E-\mu)/(kT)} \pm 1} E\, \mathrm{d}E\ . \tag{5}$$

For the nonrelativistic case, when $kT \ll mc^2$, one can neglect the term ± 1 in Eq. (3), and then the integral can be carried out exactly, yielding

$$n = g \left[\frac{mkT}{2\pi\hbar^2}\right]^{3/2} e^{(\mu - mc^2)/(kT)}\ . \tag{6}$$

The ratio of neutrons to protons is then given by

$$\frac{n_n}{n_p} = e^{-(\Delta m\, c^2 + \mu_\nu - \mu_e)/kT}\ , \tag{7}$$

where μ_ν and μ_e represent the chemical potentials for electron neutrinos and electrons, respectively, and we have ignored a factor $(m_n/m_p)^{3/2} \approx 1$. In the early universe, however, the standard theories imply that the chemical potentials for electrons and neutrinos were both negligible, so Eq. (1) above is extremely accurate.

(b) A larger Δm would mean that the Boltzmann factor described in the previous answer would be smaller, so that there would be fewer neutrons at any given temperature. Fewer neutrons implies less helium, since essentially all the neutrons that exist when the temperature falls enough for deuterium to become stable become bound into helium.

(c) There are at least four effects that occur when the electron mass/energy is taken as 1 KeV instead of 0.511 MeV:

(i) For the real mass/energy of 0.511 MeV the electron-positron pairs freeze out before nucleosynthesis, but a mass/energy of 1 KeV would mean that electron-positron pairs would behave as massless particles throughout the nucleosynthesis process. Just like adding an extra species of neutrino, this additional massless particle would mean that the expansion rate would be larger, since for a flat universe,

$$H^2 = \frac{8\pi}{3} G\rho\ ,$$

and

$$\rho = \frac{u}{c^2} = g\frac{\pi^2}{30}\frac{(kT)^4}{\hbar^3 c^5} \ .$$

Faster expansion means that the weak interactions "freeze out" earlier, since the freeze-out point is the time at which the interactions can no longer maintain equilibrium as the universe expands. An earlier freeze-out means a higher temperature of freeze-out and hence more neutrons at the time of freeze-out. In addition, the faster expansion rate means faster cooling, which means less time before the temperature of nucleosynthesis is reached, and therefore less time for neutrons to decay. Thus, faster expansion means more neutrons. Since essentially all the neutrons present when the deuterium bottleneck breaks are collected into helium, this implies more helium.

(ii) The most important reactions that keep protons and neutrons in thermal equilibrium all involve electrons and positrons:

$$n + e^+ \longleftrightarrow p + \bar{\nu}_e$$

$$n + \nu_e \longleftrightarrow p + e^- \ .$$

If the electron-positron mass/energy were smaller, then the rates of all of these reactions would be enhanced. The reactions in which an e^+ or e^- appears in the initial state will be enhanced by the presence of more e^+'s and e^-'s, and the reactions in which they appear in the final state will be enhanced because a lighter final state is easier to produce. The enhanced rate for these reactions will keep neutrons and protons in thermal equilibrium longer, and hence to lower temperatures, and this would decrease the final abundance of neutrons. Thus this effect will go in the opposite direction as effect (i), leading to the production of less helium.

(iii) If the electron mass is decreased, then the neutron decay

$$n \longrightarrow p + e^- + \bar{\nu}_e$$

becomes more exothermic, so it will happen more quickly. Thus more neutrons can decay, leading to less helium.

(iv) As mentioned in (i), lowering the mass/energy of electron-positron pairs to 1 KeV would mean that their freeze-out would not occur until after nucleosynthesis is over. In the real case, however, with $m_e c^2 = 0.511$ MeV, the electron-positron pairs start to freeze out at $t \approx 10$ sec. The energy released by this freeze-out heats the photons, protons, and neutrons, and this extra heat delays the time when the universe cools enough to break the deuterium bottleneck so that helium production can proceed. The delay allows more time for the neutrons to decay, resulting in less helium. Since the freeze-out that occurs for $m_e c^2 = 0.511$ MeV results in less helium, the absence of this freeze-out if $m_e c^2 = 1$ KeV would result in more helium.

Since the effects point in different directions, there is no easy way to know what the net effect will be. I (AHG) tried carrying out a full numerical integration, using the equations from P.J.E. Peebles, "Primordial helium abundance and the primordial fireball II," *Astrophysical Journal* **146**, 542-552 (1966). I found that the net effect of changing $m_e c^2$ to 1 KeV was to produce less helium. Apparently effects (ii) and (iii) above are the most significant. Of course I did not expect students to figure this out during the exam, although some students recalled this fact from the solutions in the Review Problems.

(d) Part (a) asked for the ratio of neutrons to protons, so its answer is

$$A = \frac{n_{\text{neutron}}}{n_{\text{proton}}} \ .$$

The fraction of the baryonic mass in neutrons is then

$$\frac{n_{\text{neutron}}}{n_B} = \frac{n_{\text{neutron}}}{n_{\text{neutron}} + n_{\text{proton}}} = \frac{\frac{n_{\text{neutron}}}{n_{\text{proton}}}}{\frac{n_{\text{neutron}}}{n_{\text{proton}}} + 1} = \frac{A}{1 + A} \ .$$

The fraction of the baryonic mass in helium is twice this number, since after nucleosynthesis essentially all neutrons are in helium, and the mass of each helium nucleus is twice the mass of the neutrons within it. Thus

$$Y = \frac{2A}{1 + A} \ .$$

This gives $\boxed{Y = 0.851.}$

PROBLEM 2: PRESSURE AND ENERGY DENSITY OF MYSTERIOUS STUFF *(25 points)*

(a) If $u \propto 1/\sqrt{V}$, then one can write

$$u(V + \Delta V) = u_0 \sqrt{\frac{V}{V + \Delta V}} \ .$$

(The above expression is proportional to $1/\sqrt{V + \Delta V}$, and reduces to $u = u_0$ when $\Delta V = 0$.) Expanding to first order in ΔV,

$$u = \frac{u_0}{\sqrt{1 + \frac{\Delta V}{V}}} = \frac{u_0}{1 + \frac{1}{2}\frac{\Delta V}{V}} = u_0 \left(1 - \frac{1}{2}\frac{\Delta V}{V}\right) \ .$$

The total energy is the energy density times the volume, so

$$U = u(V + \Delta V) = u_0 \left(1 - \frac{1}{2}\frac{\Delta V}{V}\right) V \left(1 + \frac{\Delta V}{V}\right) = U_0 \left(1 + \frac{1}{2}\frac{\Delta V}{V}\right) \ ,$$

where $U_0 = u_0 V$. Then

$$\boxed{\Delta U = \frac{1}{2}\frac{\Delta V}{V} U_0 \ .}$$

(b) The work done by the agent must be the negative of the work done by the gas, which is $p \, \Delta V$. So

$$\boxed{\Delta W = -p \, \Delta V} \ .$$

(c) The agent must supply the full change in energy, so

$$\Delta W = \Delta U = \frac{1}{2}\frac{\Delta V}{V} U_0 \ .$$

Combining this with the expression for ΔW from part (b), one sees immediately that

$$p = -\frac{1}{2}\frac{U_0}{V} = \boxed{-\frac{1}{2} u_0 \ .}$$

PROBLEM 3: AGE OF A UNIVERSE WITH MYSTERIOUS STUFF *(15 points)*

(a) The critical density ρ_c is defined as that density for which $k = 0$, where the Friedmann equation from the front of the exam implies that

$$H^2 = \frac{8\pi}{3} G\rho - \frac{kc^2}{R^2} \ .$$

Thus the critical density today is given by

$$\rho_c = \frac{3H_0^2}{8\pi G} \ .$$

The mass density today of any species X is then related to $\Omega_{X,0}$ by

$$\rho_{X,0} = \rho_c \Omega_{X,0} = \frac{3H_0^2 \Omega_{X,0}}{8\pi G} \ .$$

The total mass density today is then expressed in terms of its four components as

$$\rho_0 = \frac{3H_0^2}{8\pi G} \left[\Omega_{m,0} + \Omega_{r,0} + \Omega_{v,0} + \Omega_{\mathrm{ms},0}\right] \ .$$

But we also know how each of these contributions to the mass density scales with $x(t)$: $\rho_m \propto 1/x^3$, $\rho_r \propto 1/x^4$, $\rho_v \propto 1$, and $\rho_{\mathrm{ms}} \propto 1/\sqrt{V} \propto 1/x^{3/2}$. Inserting these factors,

$$\rho(t) = \frac{3H_0^2}{8\pi G} \left[\frac{\Omega_{m,0}}{x^3} + \frac{\Omega_{r,0}}{x^4} + \Omega_{v,0} + \frac{\Omega_{\mathrm{ms},0}}{x^{3/2}}\right] \ .$$

(b) The Friedmann equation then becomes

$$\left(\frac{\dot{x}}{x}\right)^2 = \frac{8\pi G}{3} \frac{3H_0^2}{8\pi G} \left[\frac{\Omega_{m,0}}{x^3} + \frac{\Omega_{r,0}}{x^4} + \Omega_{v,0} + \frac{\Omega_{\mathrm{ms},0}}{x^{3/2}}\right] - \frac{kc^2}{R^2} \ .$$

Defining

$$H_0^2 \Omega_{k,0} = -\frac{kc^2}{R^2(t_0)} \ ,$$

so

$$-\frac{kc^2}{R^2(t)} = -\frac{kc^2}{R^2(t_0)} \frac{1}{x^2} = \frac{H_0^2 \Omega_{k,0}}{x^2} \ ,$$

and then the Friedmann equation becomes

$$\left(\frac{\dot{x}}{x}\right)^2 = H_0^2 \left[\frac{\Omega_{m,0}}{x^3} + \frac{\Omega_{r,0}}{x^4} + \Omega_{v,0} + \frac{\Omega_{\mathrm{ms},0}}{x^{3/2}} + \frac{\Omega_{k,0}}{x^2}\right] \ .$$

Applying this equation today, when $\dot{x}/x = H_0$, one finds that

$$\boxed{\Omega_{k,0} = 1 - \Omega_{m,0} - \Omega_{r,0} - \Omega_{v,0} - \Omega_{\mathrm{ms},0}\ .}$$

Rearranging the equation for $(\dot{x}/x)^2$ above,

$$H_0\,\mathrm{d}t = \frac{\mathrm{d}x}{x\sqrt{\frac{\Omega_{m,0}}{x^3} + \frac{\Omega_{r,0}}{x^4} + \Omega_{v,0} + \frac{\Omega_{\mathrm{ms},0}}{x^{3/2}} + \frac{\Omega_{k,0}}{x^2}}}\ .$$

The age of the universe is found by integrating over the full range of x, which starts from 0 when the universe is born, and is equal to 1 today. So

$$\boxed{t_0 = \frac{1}{H_0}\int_0^1 \frac{\mathrm{d}x}{x\sqrt{\frac{\Omega_{m,0}}{x^3} + \frac{\Omega_{r,0}}{x^4} + \Omega_{v,0} + \frac{\Omega_{\mathrm{ms},0}}{x^{3/2}} + \frac{\Omega_{k,0}}{x^2}}}\ .}$$

Extra Credit for Super-Sharpies (no partial credit):

Since $\Omega_{\mathrm{tot}} < 1$, we use the Robertson-Walker open universe form

$$\mathrm{d}s^2 = -c^2\,\mathrm{d}\tau^2 = -c^2\,\mathrm{d}t^2 + R^2(t)\left\{\frac{\mathrm{d}r^2}{1+r^2} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right\}\ ,$$

where I have started with the general form from the front of the exam, and replaced k by -1. To discuss the radial transmission of light rays it is useful to define a new radial coordinate

$$r = \sinh\psi\ ,$$

so

$$\mathrm{d}r = \cosh\psi\,\mathrm{d}\psi = \sqrt{1+r^2}\,\mathrm{d}\psi\ ,$$

where I used the identity that $\cosh^2\psi = 1 + \sinh^2\psi$. The metric can then be rewritten as

$$\mathrm{d}s^2 = -c^2\,d\tau^2 = -c^2\,dt^2 + R^2(t)\left\{\mathrm{d}\psi^2 + \sinh^2\psi\,\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right\}\ .$$

Light rays then travel with $\mathrm{d}\tau^2 = 0$, so

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = \frac{c}{R(t)}\ .$$

If a light ray leaves the object at time t_e and arrives at Earth today, then it will travel an interval of ψ given by

$$\psi = \int_{t_e}^{t_0} \frac{c}{R(t')}\,\mathrm{d}t' \ .$$

We will need to know ψ, but we don't know either t_e or $R(t)$. So we need to manipulate the right-hand side of the above equation to express it in terms of things that we do know. Changing integration variables from t' to x, where $x = R(t')/R(t_0)$, one finds $\mathrm{d}x = \dot{x}\,\mathrm{d}t'$, and then

$$\psi = \int_{x_e}^{1} \frac{c}{R(t_0)} \frac{1}{x} \frac{\mathrm{d}x}{\dot{x}} \ .$$

Using $H = \dot{x}/x$,

$$\psi = \frac{c}{R(t_0)} \int_{x_e}^{1} \frac{\mathrm{d}x}{x^2 H} \ .$$

Now use the formula for $H = \dot{x}/x$ from part (b), so

$$\psi = \frac{c}{R(t_0) H_0} \int_{x_e}^{1} \frac{\mathrm{d}x}{x^2 \sqrt{\frac{\Omega_{m,0}}{x^3} + \frac{\Omega_{r,0}}{x^4} + \Omega_{v,0} + \frac{\Omega_{\mathrm{ms},0}}{x^{3/2}} + \frac{\Omega_{k,0}}{x^2}}} \ .$$

Here

$$\boxed{x_e = \frac{R(t_e)}{R(t_0)} = \frac{1}{1+z} \ ,}$$

and the coefficient in front of the integral can be evaluated using the Friedman equation for $k = -1$:

$$H_0^2 = \frac{8\pi}{3} G\rho_0 + \frac{c^2}{R^2(t_0)} = H_0^2 \Omega_0 + \frac{c^2}{R^2(t_0)} \ ,$$

so

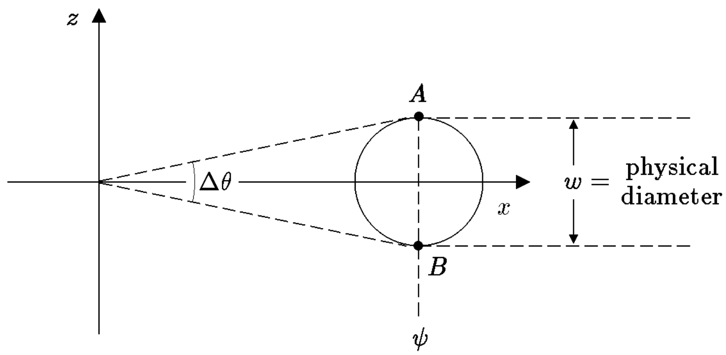
$$\frac{c^2}{R^2(t_0)\, H_0^2} = 1 - \Omega_0 = \Omega_{k,0} \ .$$

Finally, then, the expression for ψ can be written

$$\boxed{\psi = \sqrt{\Omega_{k,0}} \int_{x_e}^{1} \frac{\mathrm{d}x}{x^2 \sqrt{\frac{\Omega_{m,0}}{x^3} + \frac{\Omega_{r,0}}{x^4} + \Omega_{v,0} + \frac{\Omega_{\mathrm{ms},0}}{x^{3/2}} + \frac{\Omega_{k,0}}{x^2}}} \ ,}$$

where x_e is given by the boxed equation above.

Once we know ψ, the rest is straightforward. We draw a picture in comoving coordinates of the light rays leaving the object and arriving at Earth:



In this picture $\Delta\theta$ is the angular size that would be measured. Using the $\mathrm{d}\theta^2$ part of the metric,

$$\mathrm{d}s^2 = R^2(t)\sinh^2\psi\,\mathrm{d}\theta^2 \ ,$$

we can relate w, the physical size of the object at the time of emission, to $\Delta\theta$:

$$w = R(t_e)\sinh\psi\,\Delta\theta \ .$$

To evaluate $R(t_e)$ we can use

$$R(t_e) = x_e R(t_0) = \frac{x_e c}{H_0\sqrt{\Omega_{k,0}}} \ .$$

Finally, then,

$$\boxed{\Delta\theta = \frac{w H_0 \sqrt{\Omega_{k,0}}}{x_e c\,\sinh\psi} \ ,}$$

where ψ is given by the boxed equation above.